Magnetoconductance of parabolically confined quasi-one dimensional channels

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Electrical conduction is studied along parabolically confined quasi-one dimensional channels, in the framework of a revised linear-response theory, for elastic scattering. For zero magnetic field an explicit multichannel expression for the conductance is obtained that agrees with those of the literature. A similar but new multichannel expression is obtained in the presence of a magnetic field $B||\hat{z}$ perpendicular to the channel along the x axis. An explicit connection is made between the characteristic time for the tunnel-scattering process and the transmission and reflection coefficients that appear in either expression. As expected, for uncoupled channels the finite field expression gives the complete (Landauer-type) conductance of N parallel channels, a result that has not yet been reported in the literature. In addition, it accounts explicitly for the Hall field and the confining potential and is valid, with slight modifications, for tilted magnetic fields in the (x,z) plane.

PACS 73.40.Gk, 73.23.-b, 73.23.Ad

(February 1, 2008)

I. INTRODUCTION

The observation of the conductance quantization [1] more than a decade brought new attention to Landauer's formula [2] for the conductance of single-channel one-dimensional electronic systems and to its multichannel version derived in Ref. [3] from arguments similar to those used by Landauer. The single-channel formula [4] and a modified version of it [5] have been derived from linear-response theory. Slight variations between different results were a source of discussion [6] and crucial importance was given to the conditions of measurement. It was established that four-probe measurements do not give the same answer as the two-probe ones [7]. For a review of the subject we refer the reader to Refs. [7] and [8].

The conductance has also been studied in the presence of a magnetic field. The two-probe formula and its generalization have been found to hold. It was derived again using linear-response theory [9]. The Onsager's relation, relating the symmetry of the conductance upon changing the direction of the magnetic field, was verified. For the four-probe measurement it was realized [10] and confirmed theoretically [11] and experimentally [12] that the conductance can be asymmetric under reversal of the magnetic field.

As noted by the authors of Ref. [3] their multichannel formula does not reduce, for uncoupled channels, to that of Ref. [2]. This drawback results from their assumption that all channels originating from the reservoirs have the same electrochemical potential regardless of their velocities. In a recent Ph. D. thesis, Ref. [13], completed under the direction of one of us (CMVV), a multichannel formula, free from this drawback, has been derived for zero magnetic field.

In this work, following Ref. [13], we derive a rigorous multichannel conductance formula in the presence of a magnetic field from a revised linear-response theory. As in almost all works of the literature, it is valid for elastic scattering, i.e., in mesoscopic conductors. The formulation shows explicitly the cancellation in the product of the velocity with the quasi-one-dimensional density of states in the current carried by a channel or mode and therefore reflects some of the intuition of the original work [3]. The formula is made very explicit for parabolically confined quasi-one-dimensional channels. This type of confinement allows us to easily include the Hall field which simulates the electron-electron interaction in a mean-field sense [14]. We also consider the case of tilted magnetic fields.

In Sec. II we present a general formula for the conductivity and give the related one-electron characteristics. In Sec. III we evaluate the conductance using a scattering formulation and present various limits. Finally in Sec. IV we present a discussion of the results.

II. EXPRESSION FOR THE CONDUCTIVITY

A. New linear-response expressions

In order to explain our approach we first present some general results, in line with those from Refs. [13] and [15], which will be used to derive a general expression for the magnetoconductance. The model of the conductor or sample we use is illustrated in Fig. 1. It consists of two perfect leads (reservoirs) with random scattering centers in the middle. The longitudinal electric field representing the potential difference is applied in the inhomogenous part. A magnetic field **B** is applied along the z axis ($\vec{B} = -B\hat{z}$).

The many-body Hamiltonian that enters von Neuman's equation is

$$H_{tot}(t) = H_0 + W(t) + H^I, \tag{1}$$

where H^I represents the scattering or perturbation and $W(t)$ the external force. The free-electron part H_0 will be specified later for the geometry of Fig. 1. For elastic scattering the equation for the *many-body* density operator can be transformed to a similar one for the *one-body density* operator $\rho(t)$. The latter is the sum of the unperturbed, Fermi-Dirac operator $f(h)$ and of the perturbation operator $\tilde{\rho}(t)$, i.e., $\rho(t) = f(h) + \tilde{\rho}(t)$. For linear responses and with the initial condition $\tilde{\rho}(0) = 0$ the equation for $\tilde{\rho}(t)$ reads

$$(\partial\tilde{\rho}(t)/\partial t) + i\tilde{\mathcal{L}}\tilde{\rho}(t) = -(i/\hbar)[\tilde{w}(t), f(h)], \tag{2}$$

where $\tilde{\mathcal{L}}\bullet \equiv (1/\hbar)[h(t), \bullet]$ and $\bullet$ stands for an arbitrary one-body operator. The solution is found using the resolvent of $\tilde{\mathcal{L}}$, i.e., the Laplace transform of Eq. (2). In the Laplace domain Eq. (2) reads

$$\tilde{\rho}(s) = -\frac{i}{\hbar}\frac{1}{s + i\tilde{\mathcal{L}}}[\tilde{w}(s), f(h)] \tag{3}$$

In a representation in which H_0 is diagonal so is its one-body counterpart h_0. In this representation the operator $\tilde{\rho}$ has a diagonal ($\tilde{\rho}_d$) and a nondiagonal ($\tilde{\rho}_{nd}$) part, $\tilde{\rho} = \tilde{\rho}_d + \tilde{\rho}_{nd}$. Substituting this in Eq. (3) and acting on it with diagonal ($\mathcal{P}$) and nondiagonal ($1 - \mathcal{P}$) projection superoperators leads to two coupled equations, one for $\tilde{\rho}_d$ and one for $\tilde{\rho}_{nd}$. The steady state solution of these equations is represented by the limit $t \to \infty$. In Laplace domain this is equivalent to the limit $s \to 0+$.

The result obtained for the diagonal part $\tilde{\rho}_d$ of the density operator, the only one pertinent to the conductance, is

$$\tilde{\rho}_d = -\frac{i}{\hbar}\tilde{\Lambda}^{-1}\Gamma\sum_{\alpha\beta}[w, f(h)]_{\alpha\beta}|\psi_\alpha\rangle\langle\psi_\beta|. \tag{4}$$

Here $\tilde{\Lambda}$ and Γ are superoperators associated with the transitions caused by the perturbation h^I. They are given by $\tilde{\Lambda} = \mathcal{P}\mathcal{L}^1[1/(i\mathcal{L} + 0^+)]\mathcal{L}^1$ and $\Gamma = \mathcal{P}\,[1 - \mathcal{L}^1[1/(i\mathcal{L} + 0^+)]\mathcal{L}^1]$ with $\mathcal{L}$ and $\mathcal{L}^1$ defined by $\mathcal{L}\bullet \equiv [H, \bullet]/\hbar$ and $\mathcal{L}^1\bullet \equiv [V, \bullet]/\hbar$. $\tilde{\Lambda}$ is the one-particle scattering operator. The one-body analog w of W is related to the electric field by $e\mathbf{E}(\mathbf{r}) = -\nabla w(\mathbf{r})$. Further, $|\psi_i\rangle$ are the eigenstates of $h = h_0 + h^I$, i.e., $h|\psi_i\rangle = E_i|\psi_i\rangle$. The operator Γ doesn't affect the sum and the number $[w, f(h)]_{\alpha\beta}$. Using the relation $\Gamma|\psi_\alpha\rangle\langle\psi_\beta| = |\varphi_\alpha\rangle\langle\varphi_\beta|\delta_{\alpha\beta}$, where $|\varphi_\alpha\rangle$ is the eigenstate of h_0 and

$$\langle\psi_\alpha|[w, f(h)]|\psi_\beta\rangle = -i\hbar\,\frac{f(\epsilon_\beta) - f(\epsilon_\alpha)}{\epsilon_\beta - \epsilon_\alpha}\int_{V_0} d\mathbf{r}' E(\mathbf{r}')\langle\psi_\alpha|j(\mathbf{r}')|\psi_\beta\rangle, \tag{5}$$

with $f(h)\psi_i = f(\epsilon_i)$, we have

$$\langle\varphi_\theta|\tilde{\rho}_d|\varphi_\gamma\rangle = -\sum_{\alpha\beta}\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}f'(\epsilon_\alpha)\delta_{\alpha\beta}\int_{V_0} d\mathbf{r}'\langle\psi_\alpha|j(\mathbf{r}')|\psi_\beta\rangle E(\mathbf{r}'), \tag{6}$$

where V_0 is the volume. The current density is

$$J(\mathbf{r}) = Tr\{j(\mathbf{r})\tilde{\rho}_d\} = \sum_{\gamma\theta} \langle\varphi_\gamma|j(\mathbf{r})|\varphi_\theta\rangle\langle\varphi_\theta|\tilde{\rho}_d|\varphi_\gamma\rangle. \tag{7}$$

Substituting Eq. (6) into Eq. (7) and comparing the result with the general expression

$$J(\mathbf{r}) = \int_{V_0} d\mathbf{r}'\sigma(\mathbf{r},\mathbf{r}')E(\mathbf{r}') \tag{8}$$

we find the following expression for the conductivity

$$\sigma(\mathbf{r},\mathbf{r}') \equiv \overleftrightarrow{\sigma}_d(\mathbf{r},\mathbf{r}') = -\sum_{\gamma\theta} j_{\gamma\theta}(\mathbf{r})\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\gamma\rangle f'(\epsilon_\gamma)\langle\psi_\gamma|j(\mathbf{r}')|\psi_\gamma\rangle, \tag{9}$$

where the left-right arrow indicates that $\sigma(\mathbf{r},\mathbf{r}')$ is a tensor. The conductance G is given by

$$G = \int_A\int_{A'} dA.\overleftrightarrow{\sigma}_d(\mathbf{r},\mathbf{r}').dA' \tag{10}$$

where A and A' are two suitably chosen surfaces.

B. One-electron characteristics

Eigenfunctions and eigenvalues. We consider an electron gas which interacts only with impurities. As shown in Fig. 1, a magnetic field $\mathbf{B}$=$-B\hat{z}$ is applied along the z axis. When an electric field $\vec{E}_x$ is applied the resulting Hall field $E_\perp$ is opposite to the y axis. We consider a parabolic confining potential along the y axis, $V_y = m\Omega^2y^2/2$ and use the vector potential $\mathbf{A} = By\hat{x}$. Including the field $E_\perp$ [16] in the one-electron hamiltonian h_0 gives

$$h_0 = \frac{1}{2m}\left(\vec{P} - q\mathbf{A}\right)^2 - qE_\perp y + \frac{1}{2}m\Omega^2y^2. \tag{11}$$

We attempt a solution of Eq. (11) in the form $\varphi(x,y) = \chi(y)\exp(ik_xx)$ and introduce the variable $\xi = \hbar k_x/qB + qE_\perp/m\omega_c^2$, where $\omega_c = qB/m$ is the cyclotron frequency. Using $\omega_T^2 = \omega_c^2 + \Omega^2$, $h_0\varphi(x,y) = \epsilon\varphi(x,y)$ and completing the square we can rewrite Eq. (11) as

$$\frac{m\omega_T^2}{2}(y - \frac{\omega_c^2}{\omega_T^2}\xi)^2 - \frac{\hbar^2}{2m}\chi''(y) = E\chi(y), \tag{12}$$

where $E = \epsilon - E(k)$. With $\zeta = [y - (\omega_c^2/\omega_T^2)\xi](m\omega_T/\hbar)^{1/2}$ the solution of Eq. (12) is $\chi_n(\zeta) = e^{-\zeta^2/2}H_n(\zeta)$, where $H_n(\zeta)$ are the Hermite polynomials. The corresponding eigenvalues $\epsilon = E + E(k) \equiv \epsilon(k_x,n)$ are given by

$$\epsilon(k_x,n) = (n+1/2)\hbar\omega_T + (\hbar^2k_x^2\Omega^2 - 2\omega_C\hbar k_xqE_\perp - q^2E_\perp^2)/2m\omega_T^2, \tag{13}$$

where n is the Landau level index. From this expression we obtain the velocity $\mathbf{v} = \nabla_\mathbf{k}\epsilon(\mathbf{k})/\hbar$ along the direction of propagation. The result is

$$v_x = (\hbar k_x\Omega^2 - \omega_cqE_\perp)/m\omega_T^2. \tag{14}$$

Current density. The current density operator is expressed in terms of the one-particle eigenfunctions in matrix form. From quantum field theory [15] $\mathbf{j} = (\hbar/i)\int \Psi^* \mathbf{v} \Psi d^3r$ we obtain

$$\mathbf{j}_{\beta\alpha} = \frac{-iq\hbar}{2m}\left[\varphi_\beta^*(\nabla\varphi_\alpha) - (\nabla\varphi_\beta^*)\varphi_\alpha\right] - \frac{q}{m}\mathbf{A}\varphi_\beta^*\varphi_\alpha; \tag{15}$$

the term in the square brackets represents the standard ponderomotive or diffusion current and the term $\propto \mathbf{A}$ a deflection due to the magnetic field. We rewrite Eq. (15) in terms of the gauge-invariant derivative [9] $\mathbf{D} = \nabla - iq\mathbf{A}/m$ in the form ($f \overleftrightarrow{D} g = f\nabla g - g\nabla^* f$)

$$\mathbf{j}_{\beta\alpha} = \frac{-iq\hbar}{2m}\varphi_\beta^* \overleftrightarrow{D} \varphi_\alpha. \tag{16}$$

The current density in the x direction depends on the y coordinate. It vanishes along the y direction due to the parabolic potential confinement.

Equation (16) leads to some useful properties of the current density expressed in terms of the relevant eigenvalues. For different eigenvalues one has

$$\nabla\mathbf{j}_{\beta\alpha} = \frac{iq}{\hbar}(\epsilon_\alpha - \epsilon_\beta)\varphi_\alpha\varphi_\beta^*. \tag{17}$$

On the other hand, for eigenfunctions of the same energy the following properties hold

$$\int dy\ \overline{\varphi}_{\pm\beta}^*(\overleftrightarrow{D}\cdot\mathbf{x})\overline{\varphi}_{\pm\alpha} = \frac{\pm 2mi}{\hbar}\delta_{\alpha\beta},\ \epsilon_\beta = \epsilon_\alpha, \tag{18}$$

$$\int dy\ \overline{\varphi}_{\mp\beta}^*(\overleftrightarrow{D}\cdot\mathbf{x})\overline{\varphi}_{\pm\alpha} = 0,\ \epsilon_\beta = \epsilon_\alpha, \tag{19}$$

if the current flux is normalized instead of the eigenfunctions as shown in Ref. [9]. The new normalized eigenfunction is

$$\overline{\varphi}_{\pm,a} = e^{\pm ik_{x_a}x}\overline{\chi}_{n_a,\pm k_{x_a}}(y)/\sqrt{\theta_a}; \tag{20}$$

the normalization ($\int \overline{\chi}^2 dy = 1$) constant θ has the units of velocity; it is given by

$$\theta_{\pm a} = \left[\hbar|k_a|\Omega^2 \mp q\omega_c E_\perp\right]/m\omega_T^2 = v_{\pm a} \tag{21}$$

Notice the difference between $v_{\pm a}$, always positive, cf. Eq. (13), and the velocity given by Eq. (14).

Conductivity. In terms of the eigenfunctions of Eq. (12) the conductivity reads [13]

$$\overleftrightarrow{\sigma}_d(\mathbf{r},\mathbf{r}') = -\int f'(\epsilon_p)\overleftrightarrow{\sigma}_d^{\epsilon_p}(\mathbf{r},\mathbf{r}')d\epsilon_p, \tag{22}$$

where

$$\overleftrightarrow{\sigma}_d^{\epsilon_p}(\mathbf{r},\mathbf{r}') = \sum_s \delta(\epsilon_p - \epsilon_s)\left(\tilde{\Lambda}^{-1}j(\mathbf{r})\right)_{ss} j(\mathbf{r}')_{SS}. \tag{23}$$

Here $f'(\epsilon_s)$ is the derivative of the Fermi-Dirac function, $s \equiv \{n, k_x\}$, φ_s are the unperturbed states, and ψ_S the scattering states. We have also used the notation $\langle\varphi_s|X|\varphi_{s'}\rangle = X_{ss'}$ and $\langle\psi_S|X|\psi_{S'}\rangle = X_{SS'}$ for the matrix elements of X. The Dirac δ function is rewritten in terms of k_x using the property $\delta\left(g(k_x)\right) = \sum_i \delta(k_x - k_{x_i})/|g'(k_{x_i})|$, where g' is the derivative of $g(k_x)$ and $k_{n\pm}$ are the roots of $g(k_x) = 0$ written explicitly as

$$[\hbar^2\Omega^2 k_x^2 - 2\omega_c\hbar q E_\perp k_x - q^2 E_\perp^2]/2m\omega_T^2 + (n + 1/2)\hbar\omega_T - \epsilon_p = 0. \tag{24}$$

The roots $k_{n\pm}$ of this quadratic equation are of the form $k_{n\pm} = [-b \pm (b^2 - 4ac)^{1/2}]/2a$. They are real-and opposite to each other-if c is negative. If this condition holds the wave functions can propagate in different channels. For complex roots, the wave functions have negative exponentials and their amplitude decreases with propagation. These two roots are opposite to each other if c is negative. The propagation modes depend on confinement, magnetic field, Landau-level index, and electric field. For a given energy, $g'(k_{n\pm}) = (\hbar^2\Omega^2 k_{n\pm} - \omega_c\hbar q E_\perp)/m\omega_T^2$ and the replacement of the sum over k_x by an integral, $\sum_{k_x} \to (L/2\pi)\int_{-L/2}^{L/2} dk_x$, lead to

$$\begin{aligned} \overset{\leftrightarrow}{\sigma}{}_d^{\epsilon_p}(\mathbf{r}, \mathbf{r}') &= \sum_n^{\epsilon_p} \frac{L}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dk_x \left[\frac{\delta(k_x - k_{n+})}{|g'(k_+)|} + \frac{\delta(k_x - k_{n-})}{|g'(k_-)|}\right]\left(\tilde{\Lambda}^{-1} j(\mathbf{r})\right)_{ss} j(\mathbf{r}')_{SS} \\ &= \frac{L}{2\pi}\sum_{n_s}^{\epsilon_p}\left[M_{k_{n+}} + M_{k_{n-}}\right], \end{aligned} \tag{25}$$

where

$$M_{k_{n\pm}} = \frac{1}{|g'(k_{n\pm})|} j(\mathbf{r}')_{S\pm S\pm}\left(\tilde{\Lambda}^{-1} j(\mathbf{r})\right)_{s\pm s\pm}; \tag{26}$$

the notation $s\pm$ or $S\pm$ indicates that only the values $k_{n\pm}$ are involved in the relevant $X_{ss'}$ or $X_{SS'}$ matrix element.

III. NEW CONDUCTANCE EXPRESSIONS IN TERMS OF TRANSMISSION AND REFLECTION COEFFICIENTS

A. Scattering formulation

For clarity the two roots $k_{n\pm}$ are assumed to be in opposite directions. This holds if ac is negative and it is the case when the Hall field is neglected. Then Eqs. (10) and (23) give

$$G(\epsilon_p) = \frac{L}{2\pi}\sum_n^{\epsilon_p}\left(N_{k_{n+}} + N_{k_{n-}}\right), \tag{27}$$

where

$$N_{k_{n\pm}} = \frac{1}{|g'(k_{n\pm})|}\int dA' j(\mathbf{r}')_{S\pm S\pm}\int dA\left(\tilde{\Lambda}^{-1} j(\mathbf{r})\right)_{s\pm s\pm}. \tag{28}$$

We now proceed with the evaluation of these two integrals that are related to transmission and reflection coefficients. We can carry out the integrations by choosing two surfaces A

and A' in an asymptotic region. The choice of surface is arbitrary. It is not necessary to know the exact scattering states. It is sufficient to have their asymptotic expression in a region away from the scattering centers. The scattering states are represented by a linear combination of eigenfunctions of the unperturbed Hamiltonian. The results for the various regions are

$$\overline{\psi}_{n+} = \sum_{n'}^{\epsilon_p} t^L_{nn'}(\epsilon)\overline{\varphi}_{n'+}(\mathbf{r}), \qquad x \gg L_s, \tag{29}$$

$$\overline{\psi}_{n+} = \overline{\varphi}_{n+}(\mathbf{r}) + \sum_{n'}^{\epsilon_p} r^L_{nn'}(\epsilon_p)\overline{\varphi}_{n'-}(\mathbf{r}), \qquad x \ll 0, \tag{30}$$

$$\overline{\psi}_{n-} = \overline{\varphi}_{n-}(\mathbf{r}) + \sum_{n'}^{\epsilon_p} r^R_{nn'}(\epsilon_p)\overline{\varphi}_{n'+}(\mathbf{r}), \qquad x \gg L_s, \tag{31}$$

$$\overline{\psi}_{n-} = \sum_{n'}^{\epsilon_p} t^R_{nn'}(\epsilon_p)\overline{\varphi}_{n'-}(\mathbf{r}) \qquad x \ll 0. \tag{32}$$

Using the normalization of the flux the current density is

$$\mathbf{j}_{\beta\alpha} = \sqrt{v_\beta v_\alpha}\lambda\overline{\psi}^*_\beta \overleftrightarrow{D}\, \overline{\psi}_\alpha, \tag{33}$$

where $\lambda = -iq\hbar/2mL$. Specifically for the different regions we have

$$j_{\overline{\psi}_{n+}}(\mathbf{r}') = \lambda v_{n+} \sum_{n'}^{\epsilon_p}\sum_{n"}^{\epsilon_p} t^{L*}_{nn'} t^L_{nn"} \overline{\varphi}_{n'+} \overleftrightarrow{D}\, \overline{\varphi}_{n"+}, \qquad x \gg L_s, \tag{34}$$

$$\begin{aligned} j_{\overline{\psi}_{n+}}(\mathbf{r}') = \lambda v_{n+}\{&\overline{\varphi}^*_{n+} \overleftrightarrow{D}\, \overline{\varphi}_{n+} + \sum_{n'}^{\epsilon_p} r^{L*}_{nn'}\overline{\varphi}^*_{n'-} \overleftrightarrow{D}\, \overline{\varphi}_{n"+} \\ &+ \sum_{n''}^{\epsilon_p} r^L_{nn"}\overline{\varphi}^*_{n'+} \overleftrightarrow{D}\, \overline{\varphi}_{n"-} + \sum_{n'}^{\epsilon_p}\sum_{n''}^{\epsilon_p} r^{L*}_{nn'} r^L_{nn"}\overline{\varphi}^*_{n'-} \overleftrightarrow{D}\, \overline{\varphi}_{n"-}\}, \qquad x \ll 0, \end{aligned} \tag{35}$$

$$\begin{aligned} j_{\overline{\psi}_{n-}}(\mathbf{r}') = \lambda v_{n-}\{&\overline{\varphi}^*_{n-} \overleftrightarrow{D}\, \overline{\varphi}_{n-} + \sum_{n'}^{\epsilon_p} r^{R*}_{nn'}\overline{\varphi}^*_{n'+} \overleftrightarrow{D}\, \overline{\varphi}_{n"-} \\ &+ \sum_{n''}^{\epsilon_p} r^R_{nn"}\overline{\varphi}^*_{n'-} \overleftrightarrow{D}\, \overline{\varphi}_{n"+} + \sum_{n'}^{\epsilon_p}\sum_{n''}^{\epsilon_p} r^{R*}_{nn'} r^R_{nn"}\overline{\varphi}^*_{n'+} \overleftrightarrow{D}\, \overline{\varphi}_{n"+}\}, \qquad x \gg L_s, \end{aligned} \tag{36}$$

$$j_{\overline{\psi}_{n-}}(\mathbf{r}') = \lambda v_{n-} \sum_{n'}^{\epsilon_p}\sum_{n''}^{\epsilon_p} t^{R*}_{nn'} t^R_{nn"} \overline{\varphi}^*_{n'-}\overline{\varphi}_{n"-}, \qquad x \ll 0. \tag{37}$$

Evaluation of the first integral. Using Eqs. (18) and (19) we obtain

$$\int j_{\overline{\psi}_{n+}}(\mathbf{r}')dA' = \frac{qv_{n+}}{L}\sum_{n'}^{\epsilon_p} |t^L_{nn'}|^2, \qquad x \gg L_s, \tag{38}$$

$$\int j_{\overline{\psi}_{n+}}(\mathbf{r}')dA' = \frac{qv_{n+}}{L}\{1 - \sum_{n'}^{\epsilon_p} |r^L_{nn'}|^2\} \qquad x \ll 0, \tag{39}$$

$$\int j_{\overline{\psi}_{n-}}(\mathbf{r}')dA' = -\frac{qv_{n-}}{L}\{1 - \sum_{n'}^{\epsilon_p} |r^R_{nn'}|^2\}, \qquad x \gg L_s \tag{40}$$

$$\int j_{\overline{\psi}_{n-}}(\mathbf{r}')dA' = -\frac{qv_{n-}}{L}\sum_{n'}^{\epsilon_p} |t^R_{nn'}|^2, \qquad x \ll 0. \tag{41}$$

With flux conservation ($1 = |r|^2 + |t|^2$) we obtain the same result far away from each scattering region

$$\int j(\mathbf{r}')_{n\pm n\pm} dA' = \pm \frac{qv_{n\pm}}{L} \sum_{s'} |t_{nn'}^{L(R)}|^2 \tag{42}$$

Evaluation of the second integral. The second integral has the superoperator $\tilde{\Lambda}$. For elastic scattering it can be shown [17] that $\tilde{\Lambda}$ has an exact inverse with dimension of time (=energy/$\hbar$). We therefore write $\tilde{\Lambda} j(\mathbf{r}))_{ss} = (1/\tau_s) j_{ss}$ which leads to $(\tilde{\Lambda}^{-1} j(\mathbf{r}))_{ss} = \tau_s j_{ss}$ and

$$\int (\tilde{\Lambda} j(\mathbf{r}))_{ss} dA = \frac{1}{\tau_s} \int j_{ss} dA, \tag{43}$$

where τ_s is a characteristic time qualified below. We deduce the value of τ_s as follows. Using Eqs. (18) and (36) we have

$$\beta_\pm = \int j_{n\pm n\pm} dA = \pm(qv_{n\pm}/L). \tag{44}$$

For the integral on the left-hand side of Eq. (43), we use the result [13]

$$\left(\tilde{\Lambda} j(\mathbf{r})\right)_{n\pm n\pm} = \frac{2\pi}{\hbar} \sum_{n'} \delta(\epsilon_p - \epsilon_{n'}) |T_{n\pm n'\pm}|^2 \left(j_{n\pm n\pm} - j_{n'n'}\right), \tag{45}$$

where $T_{n\pm n'\pm} =< \varphi_{n\pm}|V|\psi_{n'\pm} >$ is the transition operator and V the scattering potential. With $(\tilde{\Lambda}^{-1} j(\mathbf{r}))_{ss} = \tau_s j_{ss}$ inspection of Eq. (45) shows that τ_s is a characteristic time associated with the tunnel-scattering process. In the following though we will refer to it simply as the characteristic time.

The Dirac δ function is rewritten in terms of the longitudinal components of the wavevector and of the two roots $k_\pm$. Then replacing the sum over $k_x^{'}$ by an integral leads to

$$\left(\tilde{\Lambda} j(\mathbf{r})\right)_{n\pm n\pm} = \frac{L}{\hbar} \sum_{n'}^{\epsilon_p} \left[\frac{|T_{n\pm n'+}|^2}{|g'(k_{n'+}^{'})|} (j_{nn} - j_{n'+}) + \frac{|T_{n\pm n'-}|^2}{|g'(k_{n'-}^{'})|} (j_{nn} - j_{n'-}) \right]. \tag{46}$$

Using Eqs. (47), (48), and (50) the characteristic time becomes

$$\frac{1}{\tau_{n\pm}} = \frac{L}{\hbar} \sum_{n'}^{\epsilon_p} \left[\frac{|T_{n\pm n'+}|^2}{|g'(k_{n'+}^{'})|} \left(1 \mp \frac{\beta_+^{'}}{\beta_{n\pm}}\right) + \frac{|T_{n\pm n'-}|^2}{|g(k_{n'-}^{'})|} \left(1 \pm \frac{\beta_-^{'}}{\beta_{n\pm}}\right) \right]. \tag{47}$$

Using Eqs. (42), (44), (47), and (28) we get

$$N_{k_{n\pm}} = \frac{q^2}{L^2} \frac{v_{n\pm}^2 \tau_{n\pm}}{|g'(k_{n\pm})|} \sum_{n'} |t_{nn'}|^2. \tag{48}$$

B. Evaluation of the conductance

Expression of $T_{ss'}$. With $V = h - h_0$ the matrix element $T_{ss'} =< \varphi_s|V|\psi_{s'} >$ of the transition operator T, between a state φ_s and a scattering state $\psi_{s'}$, becomes

$$T_{ss'} = \epsilon_{s'} < \psi_s|\varphi_{s'} > - < \varphi_s| \left(H_0|\psi_{s'} >\right). \tag{49}$$

We modify the second term on the right-hand side so that the Hamiltonian operates on the left element. In order to do so we recall the expression

$$\int \varphi^* P_x \psi dv = \int P_x(\varphi^* \psi) dv + \int (P_x^* \varphi^*)\psi dv. \tag{50}$$

With that we obtain

$$\int \varphi^* P_x(P_x\psi) dv = \int (P^2\varphi^*)\psi dv - \int \frac{\hbar^2 \partial}{\partial x}\left[\varphi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\varphi^*\right] dv. \tag{51}$$

If we combine these results with the Hamiltonian given by Eq. (11) we obtain

$$< \varphi_s|(H_0|\psi_{s'} >) = (< \varphi_s|H_0|)\psi_{s'} > -\frac{\hbar^2}{2m}\int \nabla(\varphi^* \overleftrightarrow{\nabla} \psi) dv - \frac{qB}{m}\int P_x(\varphi^* y \psi) dv. \tag{52}$$

If we combine this result with Green's theorem, we obtain

$$T_{ss'} = (\epsilon_s - \epsilon_{s'}) < \varphi_s|\psi_{s'} > +\frac{\hbar^2}{2m}\int_A d\mathbf{A}(\varphi_s^* \overleftrightarrow{\nabla} \psi_{s'}) + \frac{qB}{m}\int P_x(\varphi_s^* y \psi_{s'}) dv \tag{53}$$

The first term is zero if the energies are the same. If so, the remaining terms can be simplified. The result can be written compactly as

$$T_{ss'} = \frac{\hbar^2}{2m}\int_A d\mathbf{A}\cdot\hat{\mathbf{x}}\varphi_s^*(\overleftrightarrow{D})\psi_{s'}. \tag{54}$$

Finally, if we write it in terms of the normalized flux, we obtain

$$T_{ss'} = \frac{\sqrt{v_s v_{s'}}}{L}\frac{\hbar^2}{2m}\int_A d\mathbf{A}\cdot\hat{\mathbf{x}}\bar{\varphi}_s^*(\overleftrightarrow{D})\bar{\psi}_{s'} \tag{55}$$

$T_{ss'}$ *in terms of transmission and reflection coefficients.* To evaluate the term $T_{n\pm n'+}$, we use Eqs. (29) and (31) together with Eqs. (18) and (19). For $x \gg L_s$ we obtain

$$T_{n+n'+} = \frac{i\hbar}{L}\sqrt{v_{n+}v_{n'+}}t^L_{n'n}, \qquad T_{n-n'+} = 0. \tag{56}$$

For $x \ll 0$ the results are

$$T_{n+n'+} = \frac{i\hbar}{L}\sqrt{v_{n+}v_{n'+}}\delta_{nn'}, \qquad T_{n-n'+} = -\frac{i\hbar}{L}\sqrt{v_{n-}v_{n'+}}r^L_{n'n}. \tag{57}$$

To evaluate the term $T_{n\pm n'-}$ we use Eqs. (30) and (32) together with Eqs. (18) and (19). For $x \gg L_s$ we obtain

$$T_{n+n'-} = \frac{i\hbar}{L}\sqrt{v_{n+}v_{n'-}}r^R_{n'n}, \qquad T_{n-n'-} = -\frac{i\hbar}{L}\sqrt{v_{n-}v_{n'-}}\delta_{nn'}. \tag{58}$$

and for $x \ll 0$

$$T_{n-n'-} = -\frac{i\hbar}{L}\sqrt{v_{n-}v_{n'-}}t^R_{n'n} \qquad T_{n+n'-} = 0. \tag{59}$$

Characteristic time in terms of transmission and reflection coefficients. With the form of T and the characteristic time given by Eq. (47), the results for the various asymptotic regions are as follows. For $x \gg L_s$ we have $1/\tau_{n-} = 0$ and $1/\tau_{n+} \neq 0$. For $x \ll 0$ the results are $1/\tau_{n+} = 0$ and $1/\tau_{n-} \neq 0$. These nonzero results are given by

$$\begin{aligned}\frac{1}{\tau_{n\pm}} = \frac{\hbar}{L}\sum_{n'}^{\epsilon_p}\Big[&v_{n\pm}v_{n'+}|t^{L(R)}_{nn'}|^2(1-b_\pm)/|g'(k'_{n'+})| \\ &+ v_{n\pm}v_{n'-}|r^{R(L)}_{nn'}|^2(1+b_\pm)/|g'(k'_{n'-})|\Big]\,;\end{aligned} \tag{60}$$

here $b_\pm = \beta_{n'\pm}/\beta_n$ and + (-) corresponds to t^L, r^R (t^R, r^L). This is simplified by noticing that $g'(\mathbf{k}) = \vec{\nabla}_{\mathbf{k}}\epsilon(\mathbf{k}) = \hbar\mathbf{v}$ gives $|g'(k_{n\pm})| = \hbar v_{n\pm}$. Then Eq. (60) takes the simpler form

$$\frac{1}{\tau_{n\pm}} = \frac{1}{L}\sum_{n'}^{\epsilon_p}\left[v_{n\pm}|t^{L(R)}_{nn'}|^2(1-b_\pm) + v_{n\pm}|r^{R(L)}_{nn'}|^2(1+b_\pm)\right]. \tag{61}$$

We emphasize the importance of this result. To our knowledge, with the exception of Ref. [13] for $B = 0$, the transmission and reflection coefficients have not been associated with actual scattering time in the literature. Here, through a Master equation approach we have an *explicit* result, for finite B, relating these coefficients to the characteristic time.

Expression for the conductance Using Eqs. (48), (61) and (44) we obtain

$$N_{k_{n\pm}} = \frac{q^2 v_{n\pm}}{L|g'(k_{n\pm})|}\,\frac{\sum_{n'}|t_{nn'}|^2}{\sum_{n'}X(n,n')}, \tag{62}$$

where

$$X(n,n') = |t^{L(R)}_{nn'}|^2(1 - v_{n'+}/v_{n\pm}) + |r^{R(L)}_{nn'}|^2(1 + v_{n'-}/v_{n\pm}). \tag{63}$$

With current conservation $\sum_{n'}\left(|t^{L(R)}_{nn'}|^2 + |r^{R(L)}_{nn'}|^2\right) = 1$, this becomes

$$N_{k_{n\pm}} = \frac{q^2}{L}\sum_{n'}|t^{L(R)}_{nn'}|^2/[1 + \sum_{n'}^{\epsilon_p}Y^{RL}_\pm(n,n')], \tag{64}$$

where

$$Y^{RL}_\pm(n,n') = (|r^R_{nn'}|^2 v_{n'-} - |t^L_{nn'}|^2 v_{n'+})/v_{n\pm}. \tag{65}$$

Equations (27) and (32) give the conductance as

$$G(\epsilon_p) = \frac{q^2}{h}\sum_n^{\epsilon_p}\left[\frac{\sum_{n'}|t^L_{nn'}|^2}{\sum_{n'}Y^{RL}_+(n,n')} + \frac{\sum_{n'}|t^R_{nn'}|^2}{\sum_{n'}Y^{LR}_-(n,n')}\right] \tag{66}$$

This new conductance expression is more general than the two-terminal expressions of the literature. This can be easily appreciated by realizing that it has the following interesting features.

i) It is simplified considerably if we neglect the Hall field; then $v_{n+} = v_{n-}$ and the two terms in the square brackets become identical. The same holds in the absence of the magnetic field. Actually, for $B = 0$ Eq. (66) takes the form of Eq. (4.184) of Ref. [13]. The only difference is that in Eq. (66) the transverse channels and confining potential are *explicitly* specified whereas in Ref. [13] they are not.

ii) For uncoupled channels, i.e., for $r_{nn'} = r_{nn'}\delta_{nn'}$ and $t_{nn'} = t_{nn'}\delta_{nn'}$, Eq. (64) gives the multichannel version of Landauer's result, for identical terminals,

$$G(\epsilon_p) = \frac{q^2}{h}\sum_n^{\epsilon_p}\frac{|t_{nn}|^2}{|r_{nn}|^2} = \frac{q^2}{h}\sum_n^{\epsilon_p}\frac{T_n}{R_n} \tag{67}$$

To our knowledge this is the first expression that shows this expected [3], but absent from the literature, limit in the presence of a magnetic field.

iii) It is interesting to contrast the $B = 0$ limit of Eq. (66) with the $B = 0$ result of Ref. [3]. In this case $v_{n+} = v_{n-}$. Proceeding then as in Ref. [13] we may replace $1/v_{n'+} \propto \tau_n$ by $(1/N)\sum_{n'}(1/v_{n'})$ and make an average over the channels to obtain the result of Ref. [3], i.e.,

$$G(\epsilon_p) = \frac{q^2}{h}\sum_n T_n \frac{\sum_n(2/v_n)}{\sum_n(1 + R_n - T_n)/v_n}, \tag{68}$$

if we remember that $R_n = \sum_{n'}|r_{nn'}|^2$ and $T_n = \sum_{n'}|t_{nn'}|^2$. Despite its approximate character the procedure indicates that Eq. (66) is more general than Eq. (68) even for $B = 0$.

iv) For $R \approx 1$ and $T \ll 1$, Eq. (66) gives the standard [3], [9] result $G(\epsilon) = (q^2/h)Tr\{tt^*\}$ if we assume a *weak* [3] channel coupling such that $v_{n'} \ll v_n,\ n' < n$.

v) When the strength of the scattering is vanishingly small, we have $r \approx 0$ and $t \approx 1$. As expected, in this case for identical terminals and $v_{n'} \ll v_n, n' < n$, the conductance diverges, as realized in a four-terminal (two leads, two probes) experiment.

iv) Finally, we notice that the expression contains the Hall field, through the factors $v_{n\pm}$, cf. Eq. (21), which accounts for the electron-electron interaction in the Hartree sense [14].

C. Conductance in tilted magnetic fields

Equation (66) is valid for a perpendicular magnetic field B parallel to the z axis. It is of interest to have an expression valid for tilted fields B but the solution of Schroedinger's equation becomes very unwieldy and, to our knowledge, can be obtained only numerically when B points in an arbitrary direction. However, in one particular case a simple analytic solution exists and leads to a generalization of the conductance (66). Below we briefly derive the relevant expression since we are not aware of any pertinent result in the literature. This is the case when the field B is in the (x,z) plane and has components $B_\parallel$ along $\hat{x}$ and $B_\perp$ along $\hat{z}$. The situation is described by the vector potential $\mathbf{A} = B_\perp y\hat{x} + B_\parallel y\hat{z}$. Assuming an eigenfunction $\psi(x, y) = f(y)e^{ik_x x}$ the Hamiltonian gives

$$\left[\frac{\hbar^2 k_x^2}{2m} - y(\omega_\perp \hbar k_x + qE_\perp) + \frac{1}{2}m(\omega_B^2 + \Omega^2)y^2\right] f(y) - \frac{\hbar^2}{2m}f''(y) = \epsilon f(y) \tag{69}$$

With $\xi = (\omega_\perp \hbar k_x + qE_\perp)/m\omega_B^2$ this equation is transformed to

$$\frac{m\tilde{\omega}_T^2}{2}(y - \frac{\omega_B^2}{\tilde{\omega}_T^2}\xi)^2 f(y) - \frac{\hbar^2}{2m} f''(y) = Ef(y), \tag{70}$$

where $\tilde{\omega}_T^2 = \omega_B^2 + \Omega^2$ and $\omega_B^2 = \omega_\parallel^2 + \omega_\perp^2$. This is again an equation for a (displaced) harmonic oscillator. The corresponding eigenvalues $\epsilon \equiv \epsilon(k_x, n)$ are

$$\epsilon(k_x, n) = (n + 1/2)\hbar\tilde{\omega}_T - \left[\hbar^2 k_x^2(\Omega^2 + \omega_\parallel^2) - 2qE_\perp\omega_\perp\hbar k_x - q^2 E_\perp^2\right]/2m\omega_T^2 \tag{71}$$

As can be seen these results are similar to those obtained when the field B is parallel to the z axis. In fact, Eq. (13) can be obtained from Eq. (71) by setting $B_\parallel = 0$ which entails $\omega_B^2 = \omega_c^2$. All the analysis of Sec. III can be repeated and the result for the conductance has the same form. The only thing that changes in Eq. (66) are the roots $k_{n\pm}$, cf. Eq. (26); they now involve Eq. (71) rather than Eq. (13). As a side remark we notice that in a longitudinal magnetic field, with $B_\perp = 0$, we obtain formally the same result as in the absence of the magnetic field since the carriers are free in a parallel magnetic field.

IV. DISCUSSION

The expression for the conductance, given by Eq. (66), is very general and not limited to two identical terminals. We can interchange the indices R and T without changing the expression. This means that the conductance does not depend on the direction of the current. This and the various limits this expression reproduces show its generality.

This result for the conductance, valid when a magnetic field is present, was not anticipated in Ref. [13]. Since at first sight in a magnetic field the eigenfunctions along two opposite directions would be separated by a distance $\propto 2k_x$, it was thought that the expression would change dramatically. As shown though, incorporating directly in the one-electron Hamiltonian the magnetic field, the (parabolic) confining potential, and the Hall field, lead to an eigenfunction suitable for the calculations. It showed explicitly the cancellation in the product of the velocity with the quasi-one-dimensional density of states in the current carried by a channel or mode and simplified the final result. In addition, it allowed the consideration of tilted magnetic fields (in the (x,z) plane) and of the electron-electron interaction in a mean-field or Hartree sense since the Hall field was taken constant across the width whereas it is not since its value near the edges is different than that in the main part of the sample [14]. The last two aspects, limit ii) of Eq. (66), and Eq. (61) for the characteristic time are missing from other expressions for the magnetoconductance [3,4,9,18]. The most common general formula [9] reads $G_{mn}(\epsilon) = (q^2/h)\sum_{ac}^{\epsilon} |t_{mn,ac}|^2$, where $t_{mn,ac}$ is the transmission coefficient between channel a in terminal m and channel c in terminal n. This formula applies to a multiterminal configuration and two-probe measurements [7] whereas ours applies to a two-terminal configuration.

ACKNOWLEDGMENTS

We thank Dr. F. Benamira for fruitful discussions. This work was supported by the Canadian NSERC Grant No. OGP0121756.

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FIGURES

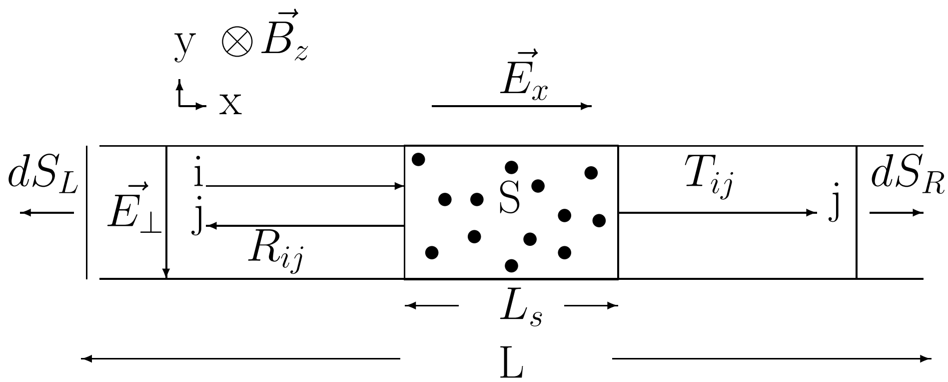

FIG. 1. A quasi-one-dimensional conductor, connected to left (L) and right (R) reservoirs in the presence of crossed electric and a magnetic fields. The length of the conductor is L. The solid dots represent random scattering centers.